B.Sc. Part II. (Infinite series) 3rd Paper

Raabe's test (contd.)

Q. Test for convergence the series

$$x^2 + \frac{2^2}{3.4}x^4 + \frac{2^2.4^2}{3.4.5.6}x^6 + \frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8 + \cdots \text{to } \infty$$

Soln Let n^{th} term of the given series $= U_n$

$$\Rightarrow U_n = \frac{2^2.4^2.6^2\ldots(2n-2)^2}{3.4.5.6\ldots 2n}x^{2n} \quad \text{---(1)}$$

Replacing n by $n+1$ in eq(1), we get

$$U_{n+1} = \frac{2^2.4^2.6^2\ldots(2n-2)^2(2n)^2}{3.4.5.6\ldots(2n)(2n+1)(2n+2)}x^{2n+2} \quad \text{---(2)}$$

Dividing (1) by (2), we get

$$\frac{U_n}{U_{n+1}} = \frac{(2n+1)(2n+2)}{(2n)^2}\cdot\frac{1}{x^2} = \frac{\left(\frac{2n+1}{n}\right)\left(\frac{2n+2}{n}\right)}{\left(\frac{2n}{n}\right)^2}\frac{1}{x^2} \quad \text{---(3)}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{4}\cdot\frac{1}{x^2}$$

$$\Rightarrow \underset{n\to\infty}{Lt}\frac{U_n}{U_{n+1}} = \frac{1}{4}\left[\underset{n\to\infty}{Lt}\frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{1}\right]\frac{1}{x^2}$$

$$= \frac{1}{4}\times 2\times 2\, x^2 = \frac{1}{x^2}$$

$$\therefore \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \frac{1}{x^2}$$

So, the given series $\sum U_n$ is convergent if $\frac{1}{x^2} > 1$
i.e. $x^2 < 1$

and $\sum U_n$ is divergent if $\frac{1}{x^2} < 1$
i.e. $x^2 > 1$.

Ratio test fails if $\frac{1}{x^2} = 1$ i.e. $x^2 = 1$.

Then, from (3)

$$\frac{U_n}{U_{n+1}} = \frac{(2n+1)(2n+2)}{(2n)^2}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} - 1 = \frac{(2n+1)(2n+2) - (2n)^2}{(2n)^2}$$

$$= \frac{\cancel{4n^2} + 2n + 4n + 2 - \cancel{4n^2}}{4n^2} = \frac{6n+2}{4n^2}$$

$$= \frac{3n+1}{2n^2}$$

$$\Rightarrow n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{n(3n+1)}{2n^2} = \frac{3n^2+n}{2n^2} = \frac{3+\frac{1}{n}}{2}$$

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\left(\frac{U_n}{U_{n+1}} - 1\right)\right] = \frac{3+0}{2} = \frac{3}{2} > 1$$

So, by Raabe's test, $\sum U_n$ is convergent if $x^2 = 1$.